

High phase-lag order trigonometrically fitted two-step Obrechhoff methods for the numerical solution of periodic initial value problems

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Abstract In this paper, we present the two-step trigonometrically fitted symmetric Obrechhoff methods with algebraic order of twelve. The method is based on the symmetric two-step Obrechhoff method, with 12 algebraic order, high phase-lag order and is constructed to solve IVPs with periodic solutions such as orbital problems. We compare the new method to some recently constructed optimized methods from the literature. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

Keywords Obrechhoff methods, Trigonometrically-fitting, Initial value problems, Symmetric multistep methods, oscillating solution.

Mathematics Subject Classification (2000) MSC 65I05 · MSC 65I07 · 65I20

1 Introduction

In this paper, the symmetric Obrechhoff methods for solving special class of initial value problems associated with second order ordinary differential equations of the type

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad (1.1)$$

in which the first order derivatives do not occur explicitly, are discussed. The numerical integration methods for (1.1) can be divided into two distinct classes:

1. Problems for which the solution period is known (even approximately) in advance.

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2. Problems for which the period is not known.

For several decades, there has been strong interest in searching for better numerical methods to integrate first-order and second-order initial value problems, because these problems are usually encountered in celestial mechanics, quantum mechanical scattering theory, theoretical physics and chemistry, and electronics. Generally, the solution of (1) is periodic, so it is expected that the result produced by some numerical methods preserves the analogical periodicity of the analytic solution [9-22]. Computational methods involving a parameter proposed by Gautschi [8], Jain et al. [10], Sommeijer and et al [30] and Steifel and Bettis [31] yield numerical solution of problems of class (1). Chawla and et al [3, 4, 5], Ananthakrishnaiah [2], Shokri and et al. [23, 24, 25, 26], Dahlquist [6], Franco [7], Lambert and Watson [9], Tsitouras and Simos [32], Simos and et al. [27, 28, 29], Hairer [9], Wang et al. [34, 35, 36], Saldanha and Achar [22], and Daele and Vanden Berghe [33] have developed methods to solve problems of class (2). Consider Obrechhoff method of the form

$$\sum_{i=0}^k \alpha_i y_{n-j+1} = \sum_{i=1}^l h^{2i} \sum_{j=0}^k \beta_{ij} y_{n-j+1}^{(2i)}, \quad (1.2)$$

for the numerical integration of the problem (1.1). The method (1.2) is symmetric when $\alpha_j = \alpha_{k-j}$, $\beta_j = \beta_{k-j}$, $j = 0, 1, 2, \dots, k$, and it is of order q if the truncation error associated with the linear difference operator is given as

$$TE = C_{q+2} h^{q+2} y^{(q+2)}, \quad x_{n-k+1} < \eta < x_{n+1},$$

where C_{q+2} is a constant dependent on h . When the method (1.2) is applied to the test problem, we get the characteristic equation as

$$\rho(\xi) - \sum_{i=1}^l (-1)^i v^{2i} \sigma_i(\xi) = 0, \quad (1.3)$$

where $v = \lambda h$ and

$$\rho(\xi) = \sum_{j=0}^k \alpha_j \xi^{k-j}, \quad \sigma_i(\xi) = \sum_{j=0}^k \beta_{ij} \xi^{k-j}, \quad i = 1, 2, \dots, l. \quad (1.4)$$

Definition 1.1 The method (1.2) is said to have interval of periodicity $(0, v_0^2)$ if for all $v^2 \in (0, v_0^2)$ the roots of Eq. (1.3) are complex and at least two of them lie on the unit circle and the others lie inside the unit circle.

Definition 1.2 The method (1.2) is said to be P-stable if its interval of periodicity is $(0, \infty)$.

Definition 1.3 For any symmetric multistep methods, the phase-lag (frequency distortion) of order q is given by

$$t(v) = v - \theta(v) = C v^{q+1} + O(v^{q+2}), \quad (1.5)$$

where C is the phase lag constant and q is the phase-lag order.

The characteristic equation of the method (1.2) is given by

$$\Omega(s : v^2) = A(v)s^2 - 2B(v)s + A(v) = 0, \quad (1.6)$$

where

$$A(v) = 1 + \sum_{i=1}^m (-1)^i \beta_{i0} v^{2i}, \quad B(v) = 1 + \sum_{i=1}^m (-1)^i \beta_{i1} v^{2i}, \quad (1.7)$$

Ψ contains polynomial functions together with trigonometric polynomials

$$\Psi_{trig} = \{1, t, \dots, t^K, \cos(r\omega t), \sin(r\omega t), \quad r = 1, 2, \dots, P\}. \quad (1.8)$$

The resulting methods are then based on a hybrid set of polynomials and trigonometric functions. If P is limited to $P = \frac{M-2}{2}$, we called method with zero phase-lag.

Remark 1.1 We present here the trigonometric versions of the set. In case ω is purely imaginary one obtains the hyperbolic description of this set. This set is characterized by two integer parameters K and P . The set in which there is no polynomial part is identified by $K = -1$ while the set in which there is no trigonometric polynomial component is identified by $K = -1$. For each problem one has $K + 2P = M - 3$, where $M - 1$ is the maximum exponent present in the full polynomial basis for the same problem.

2 Construction of the new method

From the form (1.2) and without loss of generality we assume $\alpha_j = \alpha_{m-j}$, $\beta_{i,j} = \beta_{i,m-j}$, $j = 0(1)\lfloor \frac{m}{2} \rfloor$ and we can write

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^m h^{2i} [\beta_{i0} y_{n+1}^{(2i)} + \beta_{i1} y_n^{(2i)} + \beta_{i0} y_{n-1}^{(2i)}], \quad (2.1)$$

when $m = 3$ we get

$$\begin{aligned} y_{n+1} - 2y_n + y_{n-1} = & h^2 [\beta_{10}(y_{n+1}^{(2)} + y_{n-1}^{(2)}) + \beta_{11}y_n^{(2)}] \\ & + h^4 [\beta_{20}(y_{n+1}^{(4)} + y_{n-1}^{(4)}) + \beta_{21}y_n^{(4)}] \\ & + h^6 [\beta_{30}(y_{n+1}^{(6)} + y_{n-1}^{(6)}) + \beta_{31}y_n^{(6)}]. \end{aligned} \quad (2.2)$$

$M - 3$ for method (2.2) is 11 so that if $P = -1$, $K = 13$ we obtain classic method and the coefficients of this method are

$$\begin{aligned} \beta_{1,0} &= \frac{229}{7788}, \quad \beta_{1,1} = \frac{3665}{3894}, \quad \beta_{2,0} = -\frac{1}{2360}, \\ \beta_{2,1} &= \frac{711}{12980}, \quad \beta_{3,0} = \frac{127}{39251520}, \quad \beta_{3,1} = \frac{2923}{3925152}, \end{aligned} \quad (2.3)$$

where its phase-lag is given by

$$pl_{clas} := -\frac{45469}{3394722659328000}v^{12} + O(v^{14}),$$

and its local truncation error is given by

$$LTE_{clas} = -\frac{45469}{1697361329664000}h^{14}y^{(14)} + O(h^{16}).$$

If $P = 6$, $K = -1$ we obtain the method with zero phase-lag (PL), and the coefficients of this case are given in [22].

2.1 The first formula

If $P = 0$, $K = 11$, so we called PL', we have

$$\begin{aligned} \beta_{1,0} &= \frac{1}{6v^2} \frac{\beta_{1,0num}}{A}, \quad \beta_{1,1} = \frac{1}{3v^2} \frac{\beta_{1,1num}}{A}, \quad \beta_{2,0} = \frac{-1}{5040v^2} \frac{\beta_{2,0num}}{A}, \\ \beta_{2,1} &= \frac{1}{2520v^2} \frac{\beta_{2,1num}}{A}, \quad \beta_{3,0} = \frac{-1}{10080v^2} \frac{\beta_{3,0num}}{A}, \quad \beta_{3,1} = \frac{1}{5040v^2} \frac{\beta_{3,1num}}{A}, \end{aligned} \quad (2.4)$$

where

$$A = 15120 \cos v - 15120 + 6900v^2 - 313v^4 + 660v^2 \cos v + 13v^4 \cos v,$$

and

$$\begin{aligned} \beta_{1,0num} &= -45360v^2 + 3702v^4 - 89v^6 + 78v^4 \cos v + 2v^6 \cos v + 90720 - 90720 \cos v, \\ \beta_{1,1num} &= 45360v^2 \cos v + 16998v^4 - 850v^6 + 37v^6 \cos v - 90720 + 90720 \cos v \\ &\quad + 1902v^4 \cos v, \\ \beta_{2,0num} &= -65520v^2 \cos v - 1597680v^2 + 105840v^4 - 1907v^6 + 17v^6 \cos v + 3326400 \\ &\quad - 3326400 \cos v, \\ \beta_{2,1num} &= 3109680v^2 \cos v + 14278320v^2 - 30257v^6 + 1907v^6 \cos v \\ &\quad - 34776000 + 34776000 \cos v + 105840v^4 \cos v, \\ \beta_{3,0num} &= 3360v^2 \cos v + 62160v^2 - 3814v^4 + 59v^6 + 34v^4 \cos v - 131040 + 131040 \cos v, \\ \beta_{3,1num} &= 149520v^2 \cos v + 1428000v^2 - 60514v^4 + 59v^6 \cos v - 3155040 \\ &\quad + 3155040 \cos v + 3814v^4 \cos v, \end{aligned}$$

for small values of v the above formulae are subject to heavy cancelations. In this case the following Taylor series expansion must be used:

$$\begin{aligned} \beta_{1,0} &= \frac{229}{7788} + \frac{45469}{1314147120}v^2 + \frac{85771}{341152592352}v^4 + \frac{42739761203}{29358705101073004800}v^6 \\ &\quad + \frac{3801508031029}{608197283570236453277184}v^8 + \frac{168279971604233}{13575027728788584540475136000}v^{10} \\ &\quad - \frac{266348222900207221}{2703381808485285252094734713548800}v^{12} + \dots, \end{aligned}$$

$$\beta_{1,1} = \frac{3665}{3894} - \frac{45469}{657073560}v^2 - \frac{85771}{170576296176}v^4 - \frac{42739761203}{14679352550536502400}v^6 \\ - \frac{3801508031029}{304098641785118226638592}v^8 - \frac{168279971604233}{6787513864394292270237568000}v^{10} \\ + \frac{266348222900207221}{1351690904242642626047367356774400}v^{12} + \dots,$$

$$\beta_{2,0} = -\frac{1}{2360} - \frac{45469}{30105915840}v^2 - \frac{12253}{1116499393152}v^4 - \frac{42739761203}{672581244133672473600}v^6 \\ - \frac{3801508031029}{13933246859972689656895488}v^8 - \frac{168279971604233}{310991544332247573109066752000}v^{10} \\ + \frac{266348222900207221}{61932019612571989411624831619481600}v^{12} + \dots,$$

$$\beta_{2,1} = \frac{711}{12980} - \frac{1045787}{33116507424}v^2 - \frac{1409095}{6140746662336}v^4 - \frac{983014507669}{739839368547039720960}v^6 \\ - \frac{437173423568335}{76632857729849793112925184}v^8 - \frac{3870439346897359}{342090698765472330419973427200}v^{10} \\ + \frac{266348222900207221}{2961966155383877754469013686149120}v^{12} + \dots,$$

$$\beta_{3,0} = \frac{127}{39251520} + \frac{45469}{1528454188800}v^2 + \frac{12253}{56683815344640}v^4 + \frac{42739761203}{34146432394478756352000}v^6 \\ + \frac{3801508031029}{707380225198613474888540160}v^8 + \frac{168279971604233}{15788801481483338327075696640000}v^{10} \\ - \frac{266348222900207221}{3144240995715193308590183759142912000}v^{12} + \dots,$$

$$\beta_{3,1} = \frac{2923}{3925152} - \frac{14231797}{9934952227200}v^2 - \frac{3835189}{368444799740160}v^4 - \frac{13377545256539}{221951810564111916288000}v^6 \\ - \frac{1189872013712077}{4597971463790987586775511040}v^8 - \frac{52671631112124929}{102627209629641699125992028160000}v^{10} \\ + \frac{83366993767764860173}{20437566472148756505836194434428928000}v^{12} + \dots.$$

The phase-lag and the local truncation error for the PL' method are given by

$$LTE_{PL'} = (1 - \beta_{1,1} - 2\beta_{1,0})h^2y_n^{(2)} + \left(\frac{1}{12} - \beta_{1,0} - 2\beta_{2,0} - \beta_{2,1}\right)h^4y_n^{(4)} \\ + \left(\frac{1}{360} - \frac{\beta_{1,0}}{12} - \beta_{2,0} - 2\beta_{3,0} - \beta_{3,1}\right)h^6y_n^{(6)} + \left(\frac{2}{8!} - \frac{2\beta_{1,0}}{6!} - \frac{2\beta_{2,0}}{4!} - \frac{2\beta_{3,0}}{2!}\right)h^8y_n^{(8)} \\ - \left(\frac{2}{10!} - \frac{2\beta_{1,0}}{8!} - \frac{2\beta_{2,0}}{6!} - \frac{2\beta_{3,0}}{4!}\right)h^{10}y_n^{(10)} + \left(\frac{2}{12!} - \frac{2\beta_{1,0}}{10!} - \frac{2\beta_{2,0}}{8!} - \frac{2\beta_{3,0}}{6!}\right)h^{12}y_n^{(12)} \\ + \left(\frac{2}{14!} - \frac{2\beta_{1,0}}{12!} - \frac{2\beta_{2,0}}{10!} - \frac{2\beta_{3,0}}{8!}\right)h^{14}y_n^{(14)} + O(h^{16}).$$

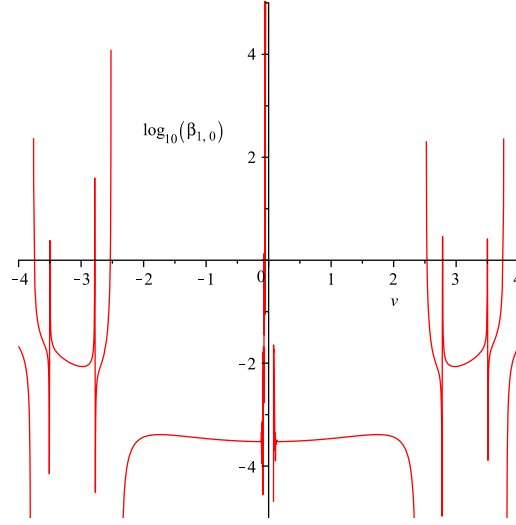


Fig. 2.1 Behavior of the coefficient $\beta_{1,0}$ in the method of PL'.

hence

$$pl_{PL'} = \frac{731602960042513638469539403}{1287287007659726361217210431335975522416459776000000} v^{24},$$

and

$$LTE_{PL'} = -\frac{45469}{1697361329664000} \left(y^{(14)} + \omega^2 y^{(12)} \right) h^{14},$$

where $v = \omega h$, ω is the frequency and h is the step length. As $v \rightarrow 0$, the LTE of the method (2.2) with derived coefficients (2.4) tends to $\frac{45469}{1697361329664000} h^{14} y^{(14)} + O(h^{16})$, which agrees with the LTE of the three methods due to Wang [36], Simos [27] and Daele [33], Achar [1], as $H \rightarrow 0$. The behavior of the coefficients of the PL' method are shown in Figures 2.1, to 2.6.

2.2 The second formula

If $P = 2$, $K = 7$, so we called PL'', we have

$$\beta_{1,0} = \frac{89}{1878} - \frac{7560}{313} \beta_{3,1}, \quad \beta_{1,1} = \frac{850}{939} + \frac{15120}{313} \beta_{3,1}, \quad \beta_{2,0} = -\frac{1907}{1577520} + \frac{330}{313} \beta_{3,1},$$

$$\beta_{2,1} = \frac{30257}{788760} + \frac{6900}{313} \beta_{3,1}, \quad \beta_{3,0} = \frac{59}{3155040} - \frac{13}{626} \beta_{3,1}, \quad \beta_{3,1} = \frac{1}{1080} \frac{A}{B},$$

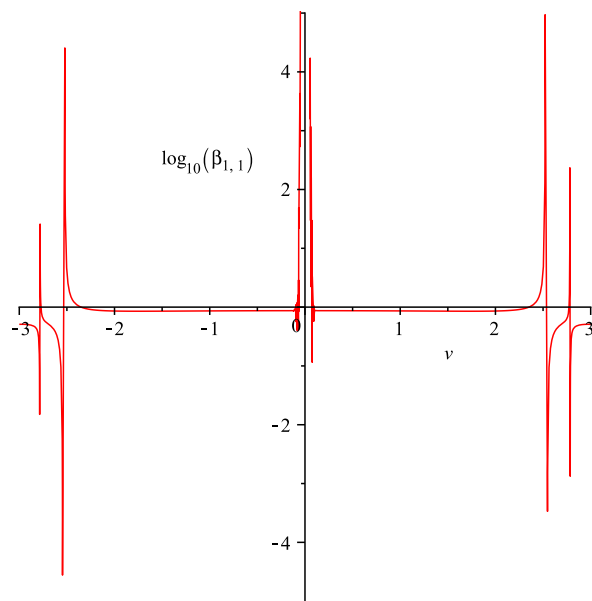


Fig. 2.2 Behavior of the coefficient $\beta_{1,1}$ in the method of PL'.

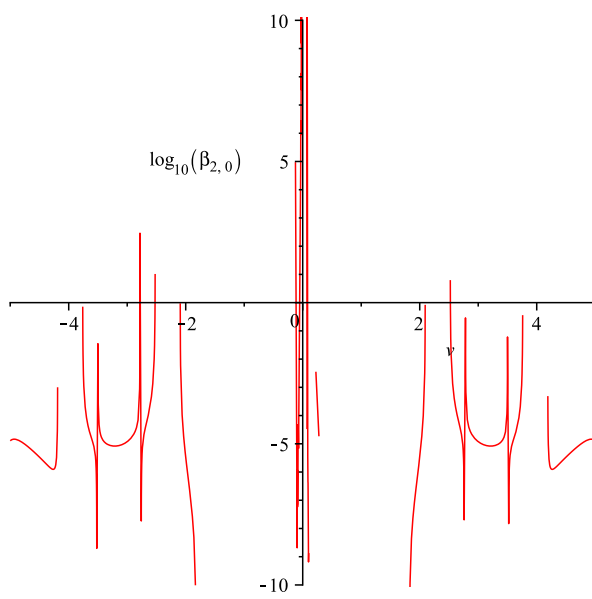


Fig. 2.3 Behavior of the coefficient $\beta_{2,0}$ in the method of PL'.

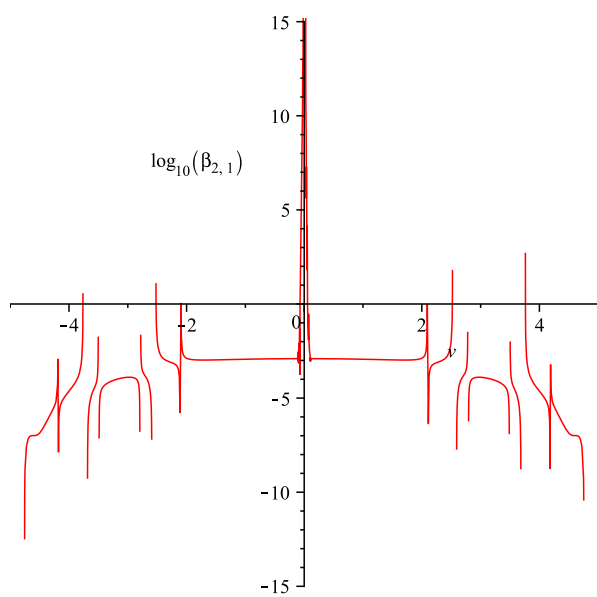


Fig. 2.4 Behavior of the coefficient $\beta_{2,1}$ in the method of PL'.

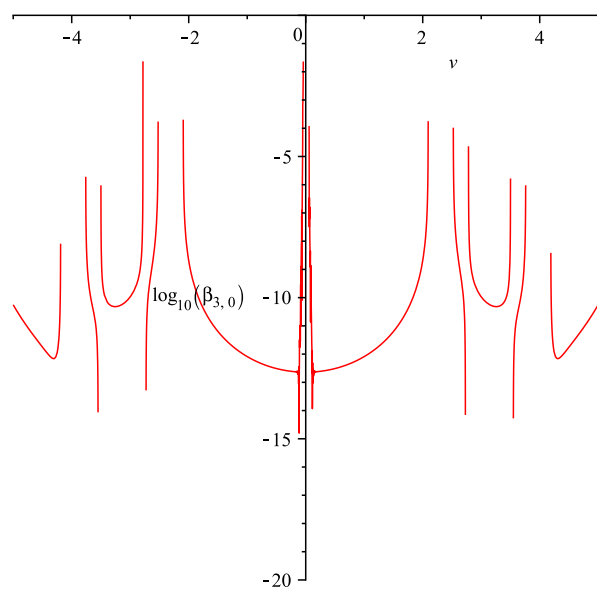


Fig. 2.5 Behavior of the coefficient $\beta_{3,0}$ in the method of PL'.

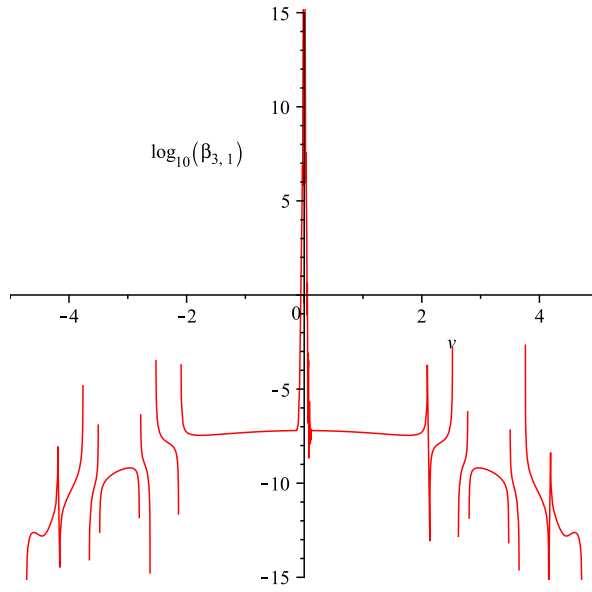


Fig. 2.6 Behavior of the coefficient $\beta_{3,1}$ in the method of PL'.

where

$$\begin{aligned}
 A = & -14400 + 213800 \cos(3v) v^4 \cos(v) - 36000 \cos(2v) \cos(v) v^2 + 14400 \cos(3v) \cos(v) \cos(2v) \\
 & - 72000 \cos(3v) \cos(v) v^2 + 20275 \cos(3v) v^4 \cos(2v) - 93600 \cos(3v) v^2 \cos(2v) \\
 & + 9660 \cos(3v) v^6 \cos(2v) + 20832 \cos(3v) v^6 \cos(v) - 10332 \cos(v) v^6 \cos(2v) + 14400 \cos(v) \\
 & - 14400 \cos(3v) \cos(2v) - 14400 \cos(2v) \cos(v) + 14400 \cos(2v) + 14400 \cos(3v) \\
 & - 116475 \cos(2v) v^4 \cos(v) + 100800 \cos(3v) \cos(v) \cos(2v) v^2 \\
 & + 29400 \cos(3v) \cos(v) \cos(2v) v^4 + 720 \cos(3v) \cos(v) \cos(2v) v^6 + 7200 \cos(v) v^2 \\
 & + 2875 \cos(v) v^4 + 1830 \cos(v) v^6 + 28800 \cos(2v) v^2 + 99200 \cos(2v) v^4 - 46848 \cos(2v) v^6 \\
 & + 64800 \cos(3v) v^2 - 249075 \cos(3v) v^4 + 88938 \cos(3v) v^6 - 14400 \cos(3v) \cos(v) \\
 & - 810 \cos(3v) v^8 \cos(2v) + 1296 \cos(3v) v^8 \cos(v) - 486 \cos(v) v^8 \cos(2v),
 \end{aligned}$$

and

$$\begin{aligned}
 B = & 240 \cos(v) - 81 \cos(2v) v^4 \cos(v) - 240 \cos(3v) \cos(v) - 240 \cos(2v) \cos(v) \\
 & + 96 \cos(3v) v^4 \cos(v) + 75 \cos(v) v^4 - 1107 \cos(2v) \cos(v) v^2 + 240 \cos(3v) \cos(v) \cos(2v) \\
 & + 115 \cos(v) v^2 + 992 \cos(3v) \cos(v) v^2 - 240 - 15 \cos(3v) v^4 \cos(2v) + 115 \cos(3v) v^2 \cos(2v) \\
 & - 240 \cos(3v) \cos(2v) - 480 \cos(2v) v^4 + 992 \cos(2v) v^2 + 405 \cos(3v) v^4 - 1107 \cos(3v) v^2 \\
 & + 240 \cos(3v) + 240 \cos(2v) v^6.
 \end{aligned}$$

For small values of v the above formulae are subject to heavy cancelations. In this case the following Taylor series expansion must be used:

$$\begin{aligned}\beta_{1,0} = & \frac{229}{7788} + \frac{318283}{657073560}v^2 + \frac{1512119}{118091281968}v^4 + \frac{22946405723893}{44038057651609507200}v^6 \\ & + \frac{18296930817563773}{651639946682396199939840}v^8 + \frac{2913158423117216376847}{1649365869047813021667729024000}v^{10} \\ & + \frac{8050460719799780764991137}{68936236116374773928415735195494400}v^{12} + \dots,\end{aligned}$$

$$\begin{aligned}\beta_{1,1} = & \frac{3665}{3894} - \frac{318283}{328536780}v^2 - \frac{1512119}{59045640984}v^4 - \frac{22946405723893}{22019028825804753600}v^6 \\ & - \frac{18296930817563773}{325819973341198099969920}v^8 - \frac{2913158423117216376847}{824682934523906510833864512000}v^{10} \\ & - \frac{8050460719799780764991137}{34468118058187386964207867597747200}v^{12} + \dots,\end{aligned}$$

$$\begin{aligned}\beta_{2,0} = & -\frac{1}{2360} - \frac{45469}{6021183168}v^2 - \frac{99714443}{586162181404800}v^4 - \frac{4808531881}{1130388645602810880}v^6 \\ & - \frac{176305401655838711}{1741655857496586207111936000}v^8 - \frac{76862259930526632407}{35266441127276874790568169676800}v^{10} \\ & - \frac{1089463503416967799081153}{26321108335343095499940553438279680000}v^{12} + \dots,\end{aligned}$$

$$\begin{aligned}\beta_{2,1} = & \frac{711}{12980} - \frac{1045787}{2365464816}v^2 - \frac{10184496007}{921111999350400}v^4 - \frac{162691107254479}{369919684273519860480}v^6 \\ & - \frac{4589005587219802631}{195491984004718859981952000}v^8 - \frac{582588392135442371849}{395847808571475125200254965760}v^{10} \\ & - \frac{2446080156637919477841851381}{25176712320762960912986616332267520000}v^{12} + \dots,\end{aligned}$$

$$\begin{aligned}\beta_{3,0} = & \frac{127}{39251520} + \frac{45469}{109175299200}v^2 + \frac{274576771}{7368895994803200}v^4 + \frac{115636672827803}{39837504460225215744000}v^6 \\ & + \frac{76494288958873853}{360908278162557895351296000}v^8 + \frac{455635060442806091167}{30449831428575009630788843520000}v^{10} \\ & + \frac{136101812396019182508073199}{131285852101792282007800655206318080000}v^{12} + \dots,\end{aligned}$$

$$\begin{aligned}\beta_{3,1} = & \frac{127}{39251520} + \frac{45469}{109175299200}v^2 + \frac{274576771}{7368895994803200}v^4 + \frac{115636672827803}{39837504460225215744000}v^6 \\ & + \frac{76494288958873853}{360908278162557895351296000}v^8 + \frac{455635060442806091167}{30449831428575009630788843520000}v^{10} \\ & + \frac{136101812396019182508073199}{131285852101792282007800655206318080000}v^{12} + \dots.\end{aligned}$$

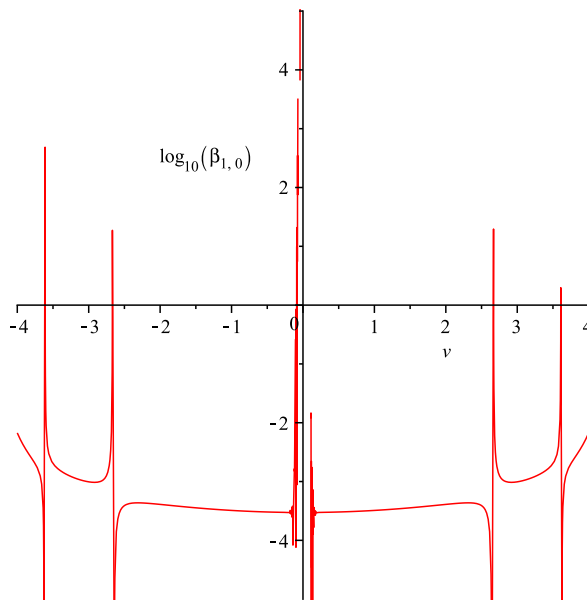


Fig. 2.7 Behavior of the coefficient $\beta_{1,0}$ in the method of PL''.

The phase-lag and the local truncation error for the PL'' method are given by

$$pl_{PL''} = -\frac{141797497314423651101}{7514399077966985427530263756800000}v^{20},$$

and

$$LTE_{PL''} = -\frac{45469h^{14}}{1697361329664000} \left(49\omega^4 y^{(10)} + y^{(14)} + 36\omega^6 y^{(8)} + 14\omega^2 y^{(12)} \right),$$

where $v = \omega h$, ω is the frequency and h is the step length. The behavior of the coefficients of the PL'' method are shown in Figures 4, 5, 6. The characteristic equation $\Omega(s : v^2) = A(v)s^2 - 2B(v)s + A(v) = 0$ has complex roots of unit magnitude when $|\cos(\theta(v))| = \left| \frac{B(v)}{A(v)} \right| < 1$, i.e. when $A(v)^2 \pm B(v)^2 > 0$. Substituting for $A(v)$ and $B(v)$ for these the two-step methods, the interval of periodicity of the classical Obrechhoff method, PL' and PL'' methods when $v \rightarrow 0$ are obtained $[0, 25.2004]$, $[0, 408.04]$ and $[0, 1428.84]$ respectively.

3 Numerical example

In this section, we present some numerical results obtained by our new two-step trigonometrically-fitted Obrechhoff methods and compare them with those from other multistep methods as

Achar: The 12th order Obrechhoff method of Achar [1].

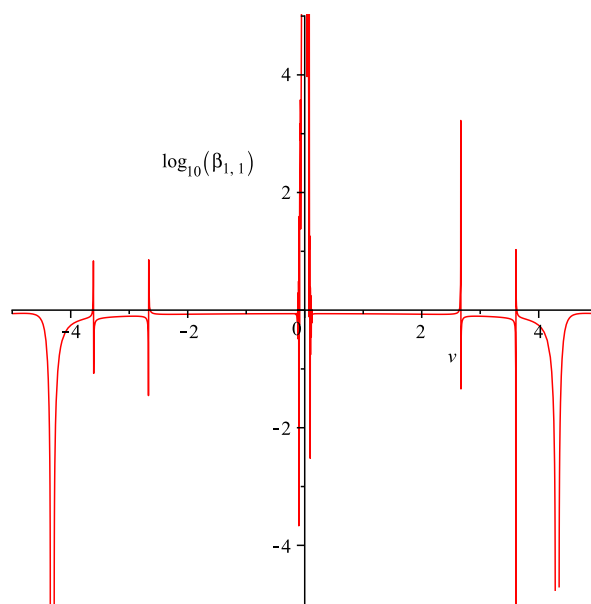


Fig. 2.8 Behavior of the coefficient $\beta_{1,1}$ in the method of PL''.

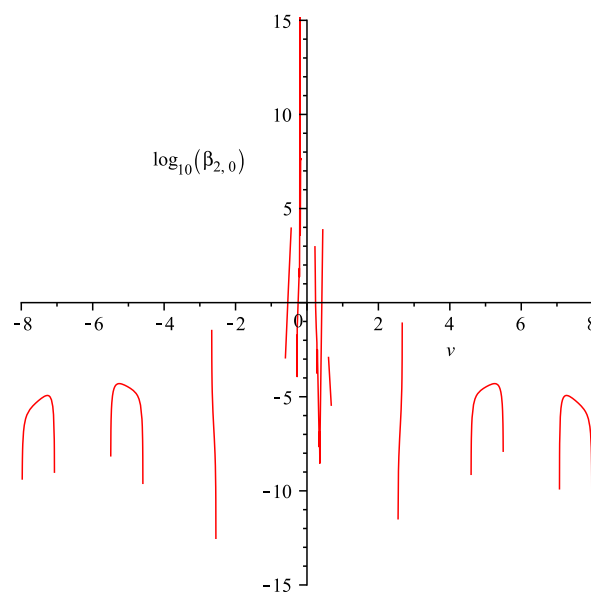


Fig. 2.9 Behavior of the coefficient $\beta_{2,0}$ in the method of PL''.

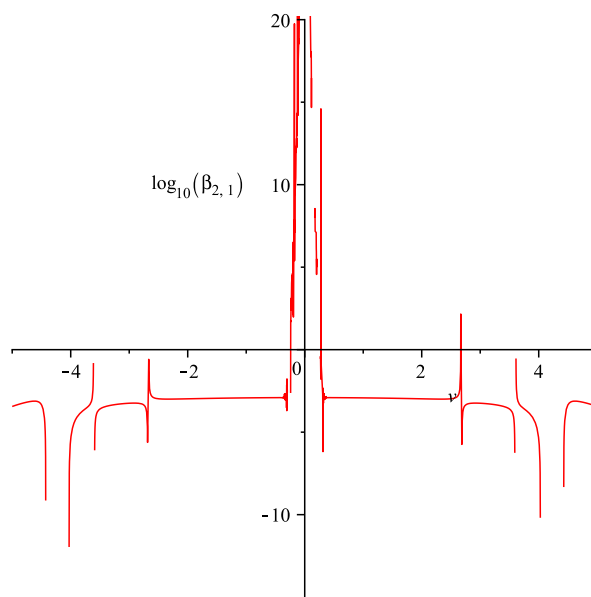


Fig. 2.10 Behavior of the coefficient $\beta_{2,1}$ in the method of PL''.

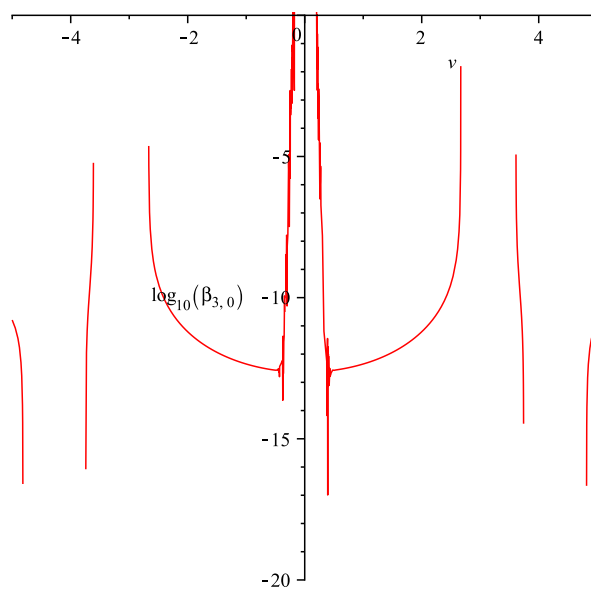


Fig. 2.11 Behavior of the coefficient $\beta_{3,0}$ in the method of PL''.

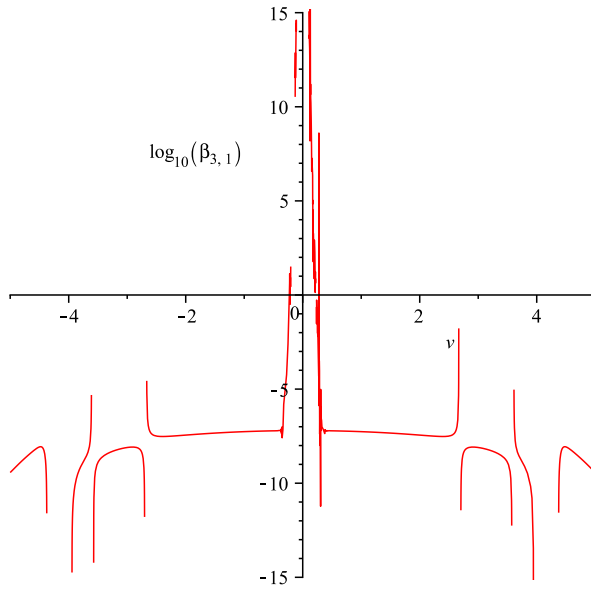


Fig. 2.12 Behavior of the coefficient $\beta_{3,1}$ in the method of PL''.

Daele: The 12th order Obrechhoff method of Van Daele [33].

Neta: The P-stable 8th-order super-implicit method of Neta [16].

Simos: The 12th order Obrechhoff method of Simos [27].

Wang: The 12th order Obrechhoff method of Wang [36].

Example 3.1 We consider the nonlinear *undamped Duffing equation*

$$y'' = -y - y^3 + B \cos(\omega x), \quad y(0) = 0.200426728067, \quad y'(0) = 0, \quad (3.1)$$

where $B = 0.002$, $\omega = 1.01$ and $x \in [0, \frac{40.5\pi}{1.01}]$. We use the following exact solution for (3.1) from [13],

$$g(x) = \sum_{i=0}^3 K_{2i+1} \cos((2i+1)\omega x),$$

where

$$\{K_1, K_3, K_5, K_7\} = \{0.200179477536, 0.246946143 \times 10^{-3}, 0.304016 \times 10^{-6}, 0.374 \times 10^{-9}\}.$$

In order to integrate this equation by a Obrechhoff method, one needs the values of y' , which occur in calculating $y^{(4)}$. These higher order derivatives can all be expressed in terms of $y(x)$ and $y'(x)$ through (3.1), i.e.

$$\begin{aligned} y^{(3)}(x) &= -(1 + 3y^2(x))y'(x) - B\omega \sin(\omega x), \\ y^{(4)}(x) &= -(1 + 3y^2(x))y''(x) - 6y(x)y'(x)^2 - B\omega^2 \cos(\omega x), \end{aligned}$$

h	PL''	Simos	Daele	Achar	Wang
$\frac{M}{500}$	6.08953e-12	3.1486e-4	4.0560e-5	4.0919e-5	4.0831e-5
$\frac{M}{1000}$	7.98859e-12	1.8069e-5	1.8733e-6	1.2708e-6	1.2678e-6
$\frac{M}{2000}$	5.52149e-12	1.0752e-6	3.8355e-8	3.9420e-8	3.9327e-8
$\frac{M}{3000}$	7.27826e-12	2.0873e-7	5.1344e-9	5.1801e-9	5.1678e-9
$\frac{M}{4000}$	6.99211e-12	6.5463e-8	3.1876e-9	1.2324e-9	1.2308e-9
$\frac{M}{5000}$	6.64542e-12	2.6673e-8	9.8900e-10	4.0911e-10	4.0741e-10

Table 3.1 Comparison of the end-point absolute error in the approximations obtained by using Methods: methods of Simos, Daele, Achar, Wang and the new method for Example 3.1.

h	PL''	Simos	Daele	Achar	Wang
$\frac{M}{500}$	1.453	1.437	1.484	1.188	1.406
$\frac{M}{1000}$	2.874	2.892	2.938	2.312	2.891
$\frac{M}{2000}$	6.267	6.233	6.36	4.812	6.236
$\frac{M}{3000}$	9.859	9.859	9.719	7.548	9.546
$\frac{M}{4000}$	13.424	13.548	13.39	9.986	13.063
$\frac{M}{5000}$	16.857	16.922	16.969	12.86	16.499

Table 3.2 CPU time for the example 3.1, are calculated for comparison among four methods: methods of Simos, Daele, Achar, Wang and our new method PL''.

x	CPU Time for PL''	PL''	Neta
2π	0.03120020	6.06453e-14	2.53e-7
4π	0.07800050	1.81249e-13	1.01e-6
6π	0.09360060	3.45171e-13	2.25e-6
8π	0.23400150	5.09481e-13	3.95e-6
10π	0.28080180	6.24098e-13	6.05e-6

Table 3.3 Comparison of the end-point absolute error in the approximations obtained by using Methods: Neta and the new method for Example 3.1.

The absolute errors at $x = \frac{40.5\pi}{1.01}$, for the new method, in comparison with methods of Simos, Daele, Achar, Wang and the new method are given in table 3.1 and the CPU times are listed in Table 3.2. Also the absolute errors at $x = 2\pi(4\pi)8\pi$, with $h = \frac{\pi}{12}$, for the new method PL'', in comparison with methods Neta and the new method are given in table 3.3.

Example 3.2 Consider the initial value problem

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11,$$

with the exact solution $y(t) = \sin(t) + \sin(10t) + \cos(10t)$. This equation has been solved numerically for $0 \leq x \leq 10\pi$ using exact starting values. In the numerical experiment, we take the step lengths $h = \pi/50, \pi/100, \pi/200, \pi/300, \pi/400$ and $\pi/500$. In Table 3.4, we present the absolute errors at the end-point and the CPU times are listed in Table 3.5.

h	PL''	Simos	Daele	Achar
$\frac{\pi}{50}$	1.76536e-26	3.0541e-11	1.2018e-11	5.7910e-13
$\frac{\pi}{100}$	4.50405e-30	2.2800e-13	7.3450e-13	5.7910e-13
$\frac{\pi}{200}$	1.90628e-34	4.3960e-13	8.6240e-13	1.3172e-12
$\frac{\pi}{300}$	4.60850e-37	2.1074e-12	2.6342e-12	1.9640e-12
$\frac{\pi}{400}$	6.28113e-39	1.3768e-12	2.9310e-12	4.7813e-12
$\frac{\pi}{500}$	2.23002e-40	6.4658e-12	2.8868e-12	7.5018e-12

Table 3.4 Comparison of the end-point absolute error in the approximations obtained by using Methods: methods of Simos, Daele, Achar and the new method for Example 3.2.

h	PL''	Simos	Daele	Achar
$\frac{\pi}{50}$	0.2652017	0.1716011	0.2496016	0.187201
$\frac{\pi}{100}$	0.5772037	0.5148033	0.5304034	0.452403
$\frac{\pi}{200}$	1.1388073	0.8580055	0.8268053	0.748805
$\frac{\pi}{300}$	1.8096116	1.1388073	1.1544074	0.951606
$\frac{\pi}{400}$	2.496016	1.3884089	1.4040091	1.23241
$\frac{\pi}{500}$	2.9484189	1.7004109	1.7784114	1.46641

Table 3.5 CPU time for the example 3.2, are calculated for comparison among four methods: methods of Simos, Daele, Achar and the new method PL''.

h	PL''	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	2.74277e-21	1.2411e-7	1.2578e-7	1.2633e-7	1.2411e-7
$\frac{4.5}{1000}$	1.54818e-24	3.8166e-9	3.9035e-9	3.8481e-9	3.8166e-9
$\frac{4.5}{2000}$	5.84727e-28	1.1931e-10	1.2288e-10	1.2002e-10	1.1931e-10
$\frac{4.5}{3000}$	5.22638e-30	1.9194e-11	2.0168e-11	1.4047e-11	1.9194e-11
$\frac{4.5}{4000}$	1.78375e-31	7.8511e-12	7.8511e-12	2.6818e-12	7.8511e-12
$\frac{4.5}{5000}$	1.28211e-32	1.6285e-12	1.6285e-12	7.4700e-14	1.6285e-12

Table 3.6 Comparison of the end-point absolute error in the approximations obtained by using five methods of Simos, Daele, Achar, Wang and the new method for Example 3.3.

Example 3.3 Consider the initial value problem

$$y'' = \frac{8y^2}{1+2x}, \quad y(0) = 1, \quad y'(0) = -2, \quad x \in [0, 4.5],$$

with the exact solution The theoretical solution of this problem is

$$y(x) = \frac{1}{1+2x}.$$

The absolute errors at $x = 4.5$ for the new method, in comparison with methods of Wang, Simos, Daele and Achar are given in the Table 3.6. The relative CPU times of computation of the new method in comparison with the other four referred methods are given in Table 3.7.

h	PL''	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	0.3588023	0.359	0.343	0.187	0.312
$\frac{4.5}{1000}$	0.6084039	0.624	0.608	0.764	1.232
$\frac{4.5}{5000}$	1.2792082	1.232	1.919	1.201	1.872
$\frac{4.5}{4000}$	1.9344124	1.888	2.590	1.622	2.558
$\frac{4.5}{5000}$	2.5584164	2.590	3.292	2.059	3.245

Table 3.7 CPU time for the example 3.3, are calculated for comparison among four methods of Simos, Daele, Achar, Wang and the new method PL''.

Conclusions

In this paper, we have presented the new trigonometrically-fitted two-step symmetric Obrechhoff methods of order 12. The details of the procedure adapted for the applications have been given in Section 2. With trigonometric fitting, we have improved the local truncation error, phase-lag error, interval of periodicity and CPU time for the classes of two-step Obrechhoff methods. The numerical results obtained by the new method for some problems show its superiority in efficiency, accuracy and stability.

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